ISRAEL JOURNAL OF MATHEMATICS 81 (1993), 289–296

FINITE COMBINATIONS OF BAIRE NUMBERS

BY

AVNER LANDVER
Department of Mathematics
The University of Kansas, Lawrence, KS 66045, USA
landver@kuhub.cc.ukans.edu

ABSTRACT

Let κ be a regular cardinal. Consider the Baire numbers of the spaces $(2^\theta)_\kappa$ for various $\theta \geq \kappa$. Let l be the number of such different Baire numbers. Models of set theory with $l = 1$ or $l = 2$ are known and it is also known that l is finite. We show here that if $\kappa > \omega$, then l could be any given finite number.

The Baire number of a topological space with no isolated points is the minimal cardinality of a family of dense open sets whose intersection is empty. The Baire number (also called the Novák number [V]) of a partial order is the minimal cardinality of a family of dense sets that has no filter [BS] (i.e. no filter on the given partial order intersecting all these dense sets non-trivially). $Fn_\kappa(\theta, 2)$ is the collection of all partial functions $p: \theta \to 2$ such that $|p| < \kappa$, and is partially ordered by reverse inclusion. For κ regular and $\theta \geq \kappa$ we consider the spaces $(2^\theta)_\kappa$ whose points are functions from θ to 2 and a typical basic open set is $\{f: \theta \to 2 \mid p \subset f\}$ where $p \in Fn_\kappa(\theta, 2)$. We denote the Baire number of $(2^\theta)_\kappa$ by $\mathfrak{n}_\kappa^\theta$. It is not hard to see that $\mathfrak{n}_\kappa^\theta$ is also the Baire number of $Fn_\kappa(\theta, 2)$. Let us now list some known facts (see [L] §1).

FACTS: *Let κ be a regular cardinal and let $\theta \geq \kappa$. Then*
1. $\kappa^+ \leq \mathfrak{n}_\kappa^\theta \leq 2^\kappa$.
2. *If $2^{<\kappa} > \kappa$, then $\mathfrak{n}_\kappa^\theta = \kappa^+$.*
3. *If $\theta_1 \leq \theta_2$, then $\mathfrak{n}_\kappa^{\theta_2} \leq \mathfrak{n}_\kappa^{\theta_1}$ and therefore $\{\mathfrak{n}_\kappa^\theta: \theta \geq \kappa \text{ is a cardinal}\}$ is finite.*
4. *If $\theta_1 \leq \theta_2$ and $\mathfrak{n}_\kappa^{\theta_2} = \theta_1$, then $\mathfrak{n}_\kappa^{\theta_1} = \theta_1$.*

Received March 9, 1992 and in revised form June 3, 1992

5. *If* $\theta = \mathfrak{n}_\kappa^{2^\kappa}$, *then* θ *is the unique cardinal with* $\mathfrak{n}_\kappa^\theta = \theta$ *and for every* $\theta_1 \geq \theta$, $\mathfrak{n}_\kappa^{\theta_1} = \theta$.

A. Miller [M] proved that $\mathrm{cof}(\mathfrak{n}_\omega^\omega) > \omega$ but also produced a model for $\mathrm{cof}(\mathfrak{n}_\omega^{\omega_1}) = \omega$. In this model $|\{\mathfrak{n}_\omega^\theta: \theta \geq \omega \text{ is a cardinal}\}| = 2$. Similar models for $\kappa > \omega$ can be found in [L]. In his above mentioned paper, Miller uses a countable support product of $Fn_\omega(\omega, 2)$ to increase $\mathfrak{n}_\omega^\omega$ without changing the value of $\mathfrak{n}_\omega^{\omega_1}$ (and hence getting $\mathfrak{n}_\omega^{\omega_1} < \mathfrak{n}_\omega^\omega$). This idea will be used next to prove the following theorem.

THEOREM: *Let* $\kappa > \omega$ *be a regular cardinal. If ZFC is consistent, then for every* $1 \leq l \in \omega$, *ZFC is consistent with* $|\{\mathfrak{n}_\kappa^\theta: \theta \geq \kappa \text{ is a cardinal}\}| = l$.

This answers ([L] 1.6) for $\kappa > \omega$. We do not know whether the Theorem is true for $\kappa = \omega$. Before we turn to the proof of the theorem, we will need the following lemma which is due to Miller. The proof of the lemma is a forcing argument that uses $\diamondsuit_\kappa$. The use of $\diamondsuit$'s in forcing arguments originated in [B]; for other such arguments see [Ka], [L] and [L1].

Definition: For the cardinals κ, θ, λ, let $Q_\kappa(\theta, \lambda)$ be the product of λ many copies of $Fn_\kappa(\theta, 2)$ with support of cardinality $\leq \kappa$. A condition $q \in Q_\kappa(\theta, \lambda)$ is a function with $\mathrm{dom}(q) \in [\lambda]^{\leq\kappa}$ and such that for every $\alpha \in \mathrm{dom}(q)$, $q(\alpha) \in Fn_\kappa(\theta, 2)$. The partial ordering is defined by putting $q \leq p$ if and only if $\mathrm{dom}(q) \supset \mathrm{dom}(p)$ and for every $\alpha \in \mathrm{dom}(p)$, $q(\alpha) \supset p(\alpha)$. If $\{q_\alpha: \alpha < \gamma\} \subset Q_\kappa(\theta, \lambda)$ have a lower bound in $Q_\kappa(\theta, \lambda)$, then let us denote the largest lower bound by $\bigwedge_{\alpha<\gamma} q_\alpha$.

LEMMA: *Let* $\kappa > \omega$ *be a regular cardinal such that* $\diamondsuit_\kappa$ *holds. Let* $\lambda, \theta \geq \kappa$ *be cardinals. Let* $Q = Q_\kappa(\theta, \lambda)$. *Then forcing with* Q *over* V *has the following property: for every function* $f: \kappa \to V$ *in the extension there is a set* $A \in V$ *such that* $(|A| = \kappa)^V$ *and* $\mathrm{range}(f) \subset A$ *(in particular, forcing with* Q *preserves* κ^+*).*

Proof of the lemma: Assume that

$$q_0 \Vdash_Q \text{“}\tau: \kappa \to V\text{”}.$$

Let M be an elementary substructure of the universe such that $|M| = \kappa$, M is closed under sequences of length $< \kappa$ (i.e. for every $\alpha \in \kappa$, ${}^\alpha M \subset M$), and such that $q_0, Q, \lambda, \theta, \kappa, \tau$ are all in M. Notice that every set in M that has cardinality $\leq \kappa$ is also a subset of M. Therefore, if $q \in Q \cap M$, then $q \subset M$.

Let $L = \{\lambda_\xi: \xi < \kappa\} = M \cap \lambda$, and $T = \{\theta_\xi: \xi < \kappa\} = M \cap \theta$. For every $\xi < \kappa$, let $L_\xi = \{\lambda_\delta: \delta < \xi\}$, and $T_\xi = \{\theta_\delta: \delta < \xi\}$. Notice that $L_\xi, T_\xi \in M$.

For every $\alpha \in \kappa$ we define a function $B_\alpha\colon Q \to \wp(\alpha \times \alpha)$ as follows:

$$(\xi,\eta) \in B_\alpha(q) \iff [\lambda_\xi \in \operatorname{dom}(q) \wedge \theta_\eta \in \operatorname{dom}(q(\lambda_\xi)) \wedge q(\lambda_\xi)(\theta_\eta) = 1].$$

Notice that for every $\alpha \in \kappa$, $B_\alpha \in M$ (because $L_\alpha, T_\alpha \in M$).

Now, let us fix a $\diamondsuit_\kappa$-sequence $I = \{I_\xi\colon \xi < \kappa\}$ on $\kappa \times \kappa$. Notice that for every $\xi < \kappa$, $I_\xi \in M$. We are now ready to construct a decreasing sequence $\{q_\alpha\colon \alpha < \kappa\} \subset Q \cap M$, below q_0, that satisfies the following conditions:

(1) $\alpha < \beta \implies q_\beta \le q_\alpha$.
(2) $(\forall \alpha < \kappa)\ L_\alpha \subseteq \operatorname{dom}(q_\alpha)$.
(3) $\alpha < \beta \implies q_\beta \restriction L_\alpha = q_\alpha \restriction L_\alpha$.
(4) If $\alpha \in \kappa$ is a limit ordinal, then $q_\alpha = \bigwedge_{\beta<\alpha} q_\beta$. (Notice that $q_\alpha \in Q \cap M$ because $\{q_\beta\colon \beta < \alpha\} \in M$.)
(5) Given q_α let us define $q_{\alpha+1}$.
Case (i): There exist $r \le q_\alpha$ such that for every $\xi < \alpha$, $\operatorname{dom}(r(\lambda_\xi)) = T_\alpha$, and $B_\alpha(r) = I_\alpha$, and r decides $\tau \restriction \alpha$. In this case, the same is true in M. Hence there are $r_\alpha, t_\alpha \in M$ such that $r_\alpha \le q_\alpha$, and for every $\xi < \alpha$, $\operatorname{dom}(r_\alpha(\lambda_\xi)) = T_\alpha$, and $B_\alpha(r_\alpha) = I_\alpha$, and

$$r_\alpha \Vdash_Q \text{“}\tau \restriction \alpha = t_\alpha\text{”}.$$

Let $q_{\alpha+1}$ be defined as follows: $q_{\alpha+1} = (q_\alpha \restriction L_\alpha) \cup (r_\alpha \restriction (\operatorname{dom}(r_\alpha) \setminus L_\alpha))$.
Case (ii): $\neg$ (case (i)). Let $q_{\alpha+1} \le q_\alpha$ be any extension in M that satisfies (2) and (3), and let $t_\alpha = \emptyset$.

Finally, define $q = \bigwedge_{\alpha<\kappa} q_\alpha$. By (1) and (3) of the construction, $q \in Q$. By (2), $\operatorname{dom}(q) = L$. Let $A = \bigcup_{\alpha<\kappa} \operatorname{range}(t_\alpha)$. We claim that

$$q \Vdash_Q \text{“}\operatorname{range}(\tau) \subset A\text{”}.$$

Assume not. Let $s \le q$, and $\delta \in \kappa$ be such that $s \Vdash_Q$ “$\tau(\delta) \notin A$”. Let us define a decreasing sequence $\{s_\alpha\colon \alpha < \kappa\}$ in Q that satisfies the following conditions:

(1) $s_0 = s$.
(2) $(\forall \alpha < \kappa)\ s_\alpha$ decides $\tau \restriction \alpha$.
(3) If $\alpha < \kappa$ is a limit ordinal, then $s_\alpha = \bigwedge_{\beta<\alpha} s_\beta$.
(4) $(\forall \alpha < \kappa)(\forall \xi < \kappa)\ T_\alpha \subset \operatorname{dom}(s_\alpha(\lambda_\xi))$.

Now let $B = \{(\xi,\eta) \in \kappa \times \kappa \colon s_{\eta+1}(\lambda_\xi)(\theta_\eta) = 1\}$. Notice that for every $\alpha < \kappa$, $B \cap \alpha \times \alpha = B_\alpha(s_\alpha)$. Let $C = \{\alpha < \kappa \colon (\forall \xi < \alpha)\ \mathrm{dom}(s_\alpha(\lambda_\xi)) = T_\alpha\}$; C is a club. In addition, $S = \{\alpha < \kappa \colon B \cap \alpha \times \alpha = I_\alpha\}$ is stationary. Pick $\alpha \in C \cap S$ such that $\alpha > \delta$. Then s_α witnesses that case (i) of part (5) in the construction of $\{q_\alpha \colon \alpha < \kappa\}$ holds (i.e. $r = s_\alpha$). So, we are given $r_\alpha, t_\alpha \in M$ such that $r_\alpha \leq q_\alpha$, and $r_\alpha \Vdash_Q$ "$\tau \restriction \alpha = t_\alpha$". Hence

$$r_\alpha \Vdash_Q \text{“}\tau(\delta) \in A\text{”}.$$

But $s_\alpha \leq r_\alpha$, and $s_\alpha \leq s$, and this implies the desired contradiction. ∎

Proof of the theorem: Since the theorem is trivial for $l = 1$, let us assume that $l \geq 2$. Start with a model V of ZFC + GCH + $\diamondsuit_\kappa$. Let

$$\kappa \leq \theta_1 < \theta_2 < \cdots < \theta_l$$

be cardinals with $\theta_i \neq \kappa^+$, and $\theta_l = \theta_{l-1}^+$, and such that if $\theta_i \neq \kappa$, then $\mathrm{cof}(\theta_i) > \kappa$. Let

$$\lambda_1 > \lambda_2 > \cdots > \lambda_l = \theta_l$$

be cardinals with $\lambda_1 = \lambda_2^+$ and such that $\mathrm{cof}(\lambda_i) > \kappa^+$.

Let $Q_i = Q_\kappa(\theta_i, \lambda_i)$. Let us force with

$$P = Q_1 \times \cdots \times Q_{l-1}.$$

By the GCH, the partial orders $Fn_\kappa(\theta_i, 2)$ all have the κ^+.c.c. ([K] VII 6.10). Therefore, P is (isomorphic to) a product of κ^+.c.c. partial orders with support of size $\leq \kappa$. Now use a delta system lemma and the Erdös–Rado theorem ($(2^\kappa)^+ \to (\kappa^+)^2_\kappa$) to show that P is κ^{++}.c.c. ([K] VIII(B7)), and hence P preserves cardinals $\geq \kappa^{++}$. Clearly, P is κ-closed and therefore cardinals $\leq \kappa$ are preserved. Finally, by the Lemma, κ^+ is preserved as well.

Let G be a P-generic filter over V. Let $\theta \neq \kappa^+$ be a cardinal with $\kappa \leq \theta \leq \theta_l$. Let i be the minimal such that $\theta \leq \theta_i$. Let us show that

$$(*) \qquad \mathfrak{n}^\theta_\kappa = \lambda_i.$$

Notice that $(*)$ suffices for the proof of the theorem since it in particular shows that $\mathfrak{n}^{\theta_l}_\kappa = \theta_l$ and therefore by fact 5, $(*)$ implies that

$$(\forall \theta \geq \theta_l)\ \mathfrak{n}^\theta_\kappa = \lambda_l.$$

In the remaining case where $\theta = \kappa^+$, $(*)$ implies that $\mathfrak{n}_\kappa^{\kappa^+} = \lambda_1$ or $\mathfrak{n}_\kappa^{\kappa^+} = \lambda_2$. Therefore, $(*)$ implies that $\{\mathfrak{n}_\kappa^\theta : \theta \geq \kappa \text{ is a cardinal}\} = \{\lambda_i : 1 \leq i \leq l\}$.

Let us first show that $\mathfrak{n}_\kappa^\theta \geq \lambda_i$. By fact 4, we may assume that $1 \leq i < l$. Notice that since P is κ-closed, $Fn_\kappa(\theta, 2)$ is absolute and has cardinality $\theta^{<\kappa} \leq \theta_i < \lambda_i$. By the product lemma, we may view forcing with P as forcing with the product $\prod\{Q_j : 1 \leq j < l \text{ and } j \neq i\} \times Q_i$. Now, by the definition of Q_i and since $\theta \leq \theta_i$, it is easy to see that any collection of $< \lambda_i$ many dense subsets of $Fn_\kappa(\theta, 2)$ in $V[G]$, has a filter.

Finally we show that $\mathfrak{n}_\kappa^\theta \leq \lambda_i$. Notice that if $i = 1$, then this is clear because $(2^\kappa = \lambda_1)^{V[G]}$ (to see this use a counting nice names argument ([K] VII)). So let us assume that $i > 1$ and hence $\theta \geq \kappa^{++}$. In addition we may assume that θ is regular (otherwise, if θ is singular, then it suffices to prove that $\mathfrak{n}_\kappa^{\theta_{i-1}^{++}} \leq \lambda_i$ since $\mathfrak{n}_\kappa^\theta \leq \mathfrak{n}_\kappa^{\theta_{i-1}^{++}}$).

Let us now view forcing with P as forcing with $S \times R$, where

$$S = Q_i \times \cdots \times Q_{l-1}$$

and

$$R = Q_1 \times \cdots \times Q_{i-1}.$$

Notice that if $i = l$, then $R = P$ and S is the trivial partial order. Let H be an S-generic filter over V, and K be an R-generic filter over $V[H]$ such that $V[H \times K] = V[G]$. For every $a : \theta \to 2$ with $|a| = \kappa$ let us define

$$D_a = \{t \in Fn_\kappa(\theta, 2) : (\exists \xi \in \operatorname{dom}(a))\ t(\xi) \neq a(\xi)\}.$$

In $V[H]$, define $\mathcal{D} = \{D_a \mid a : \theta \to 2 \text{ and } |a| = \kappa\}$. $\mathcal{D}$ is a collection of dense subsets of $Fn_\kappa(\theta, 2)$ and $|\mathcal{D}| = \lambda_i$ (because $(2^\kappa = \theta^\kappa = \lambda_i)^{V[H]}$). Let us show that $\mathcal{D}$ has no filter in $V[G]$.

Assume, by way of contradiction, that $F \in V[G]$ is a filter for $\mathcal{D}$. Assume without loss of generality that

$$\Vdash_{S \times R} \text{“}F \text{ is a filter for } \mathcal{D}\text{”}.$$

Let τ be a P-name for $\bigcup F$. It suffices to find $(s, r) \in S \times R$ and an S-name π such that

$$s \Vdash_S \text{“}[\pi : \theta \to 2 \text{ and } |\pi| = \kappa \text{ and } r \Vdash_R \text{“}\pi \subset \tau\text{”}]\text{”}.$$

We now work in V. For every $\xi \in \theta$, let $(s_\xi, r_\xi) \in S \times R$ and $u_\xi \in 2$ be such that

$$(s_\xi, r_\xi) \Vdash \text{“}\tau(\xi) = u_\xi\text{”}.$$

Consider $\{r_\xi \colon \xi \in \theta\}$. Since $\theta \geq \kappa^{++}$ and θ is regular, we may use the delta system lemma to get $X \in [\theta]^\theta$ such that $\{\mathrm{dom}(r_\xi) \colon \xi \in X\}$ form a delta system with a root Δ. Now, since $|Fn_\kappa(\theta_{i-1}, 2)| = \theta_{i-1} < \theta$ and $|\Delta| \leq \kappa$, there exists $Y \in [X]^\theta$ such that $\{r_\xi \colon \xi \in Y\}$ all agree on Δ (i.e.$(\forall \xi, \eta \in Y)\ r_\xi \restriction \Delta = r_\eta \restriction \Delta$).

Consider $\{s_\xi \colon \xi \in Y\}$. Since S is κ^{++}.c.c. there exists $s' \in S$ and a name σ with

$$s' \Vdash_S \text{“}\sigma = \{\xi \in Y \colon s_\xi \in \Gamma\} \textit{ and } |\sigma| = \theta\text{”},$$

where Γ is the canonical name for the S-generic filter. By the Lemma, there exists $A \in [Y]^\kappa$ and $s \leq s'$ such that

$$s \Vdash_S \text{“}|\sigma \cap A| = \kappa\text{”}.$$

Let π be an S-name for the function whose domain is $\sigma \cap A$ and such that for every $\xi \in \sigma \cap A$, $\pi(\xi) = u_\xi$. Let $r = \bigcup\{r_\xi \colon \xi \in A\}$. Then $r \in R$ (because $A \subset Y$ and $A \in V$), and

$$s \Vdash_S \text{“}[\pi \colon \theta \to 2, \text{ and } |\pi| = \kappa, \text{ and } r \Vdash_R \text{“}\pi \subset \tau\text{”}]\text{”}. \quad \blacksquare$$

Remark 1: If $\kappa = \omega$, then it is known that P (defined as in the proof of the Theorem but for $\kappa = \omega$) collapses ω_1 ([K] VIII(E4) and [M] p. 280), and (assuming CH) is $\aleph_2$*.c.c.* What one needs in order to get the argument of the Theorem to go through for the case $\kappa = \omega$, is the following: if σ is a set in the extension that is unbounded in $(\omega_2)^V$, then there exists a countable set A in V such that $A \cap \sigma$ is infinite. This is false by the following Proposition.

PROPOSITION: *Let $\lambda \geq \omega$, and $\theta > \omega$ be cardinals. Let $Q = Q_\omega(\theta, \lambda)$. Then forcing with Q adds a set $\sigma \subset \theta$, that is unbounded in θ, and such that if A is a countable (in V) ground model subset of θ, then $A \cap \sigma$ is finite.*

Proof: For every $n \in \omega$, let g_n be the n'th generic function (i.e. $g_n \colon \theta \to 2$, and $g_n(\alpha) = 1$ if and only if there exists p in the Q-generic filter such that $p(n)(\alpha) = 1$). Let σ be the set defined in the extension by $\sigma = \{\alpha \in \theta \colon (\forall n \in \omega)\ g_n(\alpha) = 1\}$. Since $\theta \geq (\omega_1)^V$, and the supports of members of Q are countable, it is not hard to see that σ is unbounded in θ. Now let $p \in Q$, and $A \in [\theta]^{\aleph_0}$. Let us find $q \leq p$ such that $q \Vdash$ "$|A \cap \sigma| < \aleph_0$". We may assume that $\mathrm{dom}(p) \supset \omega$.

Let $A^* = \{\alpha \in A\colon (\exists n \in \omega)\ \alpha \notin \mathrm{dom}(p(n))\}$. Notice that $A \setminus A^*$ is finite. For every $K \in [\omega]^{<\aleph_0}$ define $a(K) = \{\alpha \in A^*\colon (\forall n \notin K)\ \alpha \in \mathrm{dom}(p(n))\}$; $a(K)$ is finite. Fix $\{\alpha_i\colon i \in \omega\}$ an enumeration of A^*.

We now construct $\{q_i\colon i \in \omega\} \subset Q$, $\{n_i\colon i \in \omega\} \subset \omega$, and $\{F_i\colon i \in \omega\}$ finite subsets of A^* that satisfy the following conditions:

(1) $q_0 \leq p$ and for every $i \in \omega$, $q_{i+1} \leq q_i$.
(2) For every $i \in \omega$, $q_i \restriction (\lambda \setminus \{n_k\colon k \leq i\}) = p \restriction (\lambda \setminus \{n_k\colon k \leq i\})$.
(3) For every $i \in \omega$, $F_i \subset F_{i+1}$, and $F_i \supset a(\{n_k\colon k \leq i\})$.
(4) $\bigcup_{i \in \omega} F_i = A^*$.
(5) $i < j \implies q_j \restriction \{n_k\colon k \leq i\} = q_i \restriction \{n_k\colon k \leq i\}$.
(6) For every $i \in \omega$ and every $\alpha \in F_i$, $q_i \Vdash$ "$\alpha \notin \sigma$".

STAGE 0: Pick $n_0 \in \omega$ with $\alpha_0 \notin \mathrm{dom}(p(n_0))$. Let $F_0 = a(\{n_0\}) \cup \{\alpha_0\}$. Define $q_0(n_0)$ by:

$$q_0(n_0)(\alpha) = \begin{cases} 0 & \alpha \in F_0 \\ p(n_0)(\alpha) & \alpha \notin F_0 \text{ and } \alpha \in \mathrm{dom}(p(n_0)). \end{cases}$$

STAGE $i+1$: If $\alpha_{i+1} \in F_i$, then $n_{i+1} = n_i, F_{i+1} = F_i$, and $q_{i+1} = q_i$. Otherwise, by (3), $\alpha_{i+1} \notin a(\{n_k\colon k \leq i\})$. Therefore, we can pick $n_{i+1} \notin \{n_k\colon k \leq i\}$ such that $\alpha_{i+1} \notin \mathrm{dom}(p(n_{i+1}))$. By (2), $\alpha_{i+1} \notin \mathrm{dom}(q_i(n_{i+1}))$ as well. Let $F_{i+1} = F_i \cup a(\{n_k\colon k \leq i+1\}) \cup \{\alpha_{i+1}\}$. Define $q_{i+1}(n_{i+1})$ by:

$$q_{i+1}(n_{i+1})(\alpha) = \begin{cases} 0 & \alpha \in F_{i+1} \setminus F_i \\ q_i(n_{i+1})(\alpha) & \alpha \notin F_{i+1} \setminus F_i \text{ and } \alpha \in \mathrm{dom}(q_i(n_{i+1})). \end{cases}$$

Notice that $\alpha \in F_{i+1} \setminus F_i$ implies that either $\alpha = \alpha_{i+1}$, or $\alpha \in a(\{n_k\colon k \leq i+1\}) \setminus a(\{n_k\colon k \leq i\})$, and in either of these cases $\alpha \notin \mathrm{dom}(q_i(n_{i+1}))$.

Finally, let $q = \bigwedge_{i \in \omega} q_i$. By (2) and (5), $q \in Q$ and clearly, $q \leq p$. By (4) and (6), $q \Vdash$ "$A^* \cap \sigma = \emptyset$". ∎

Remark 2: In the extension of the above Proposition we also have: σ is an unbounded subset of θ, and if $x \in [\sigma]^{\aleph_0}$, then $(\omega_1)^V$ is countable in $V[x]$. This is true because Q is $\aleph_2$.*c.c.*, and thus there is $X \in V$ with $|X| = \aleph_1$ and $X \supset x$. Now one can enumerate X, in V, in type $(\omega_1)^V$, and x must be unbounded in this enumeration since otherwise it would be contained in a countable ground model set.

Finally, we would like to mention that the Lemma implies that, the Proposition, stated for $\kappa > \omega$ (rather than ω), is false.

References

[B] J. Baumgartner, *Almost-disjoint sets, the dense set problem, and partition calculus,* Ann. Math. Logic **10** (1976), 401–439.

[BS] B. Balcar and P. Simon, *Handbook of Boolean Algebras* (J.D. Monk and R. Bonnet, eds.), Vol. 2, North-Holland, Amsterdam, 1989, pp. 333–386.

[Ka] A. Kanamori, *Perfect-set forcing for uncountable cardinals,* Ann. Math. Logic **19** (1980), 97–114.

[K] K. Kunen, *Set Theory,* North-Holland, Amsterdam, 1980.

[L] A. Landver, *Baire numbers uncountable Cohen sets and perfect-set forcing,* J. Symbolic Logic **57** (1992), 1086–1107.

[L1] A. Landver, *Singular Baire Numbers and Related Topics,* Ph.D. Thesis, The University of Wisconsin, Madison, 1990.

[M] A. Miller, *The Baire Category Theorem and cardinals of countable cofinality,* J. Symbolic Logic **47** (1982), 275–288.

[V] B. Veličković, *Jensen's □ principles and the Novák number of partially ordered sets,* J. Symbolic Logic **51** (1986), 47–58.